Synchrotron Radiation

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Chapter 1 is for users. 重點在光子
Chapter 2 is for accelerator physicists. 重點在粒子

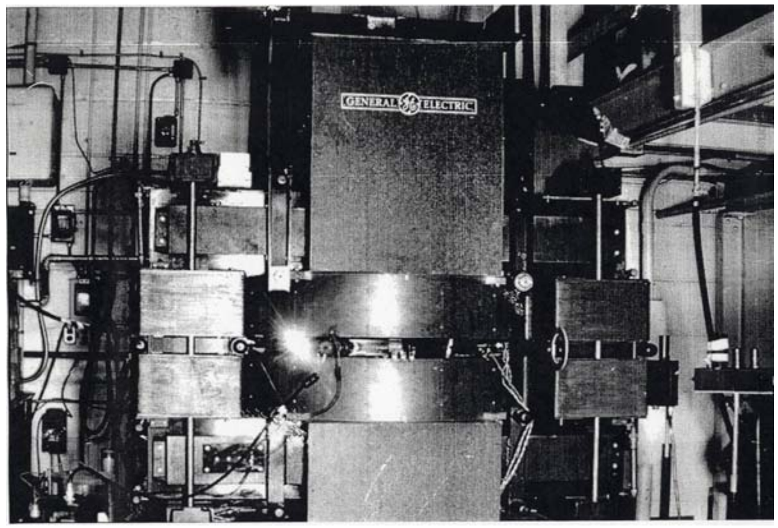
History

1883 Maxwell equations

$$\nabla \cdot \vec{B} = 0$$
$$\nabla \cdot \vec{D} = \rho$$
$$\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}$$
$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

1887 Hertz observed E&M wave
1898 Lienard-Wiechert potential
1912 Schott′s synchrotron radiation classical analysis
1946 Schwinger′s synchrotron raditation quantum analysis

1947 Synchrotron radiation observed at 70-MeV General Electric electron synchrotron. *



1970′s First generation of synchrotron radiation facilities
1980′s Second generation of synchrotron radiation facilities
1990′s Third generation of synchrotron radiation facilities
2010′s Fourth generation of synchrotron radiation facilities

*People expected E&M wave radiated. But they thought the radiation would be microwaves $\sim$ revolution frequency of the beam. Imagine the shock when they observed visible light!

History of synchrotron radiation did not at all come by as easily as implied by the textbooks. 不要被教科書騙了。

It took $\sim$ 100 years to evolve, and none of the following steps were easy:

- Maxwell equations
- electromagnetic waves
- concept of point charge
  一個實體的粒子能沒有大小嗎？
  這裡藏了一段吵翻天的物理史。
- synchrotron radiation
- 每一步都充滿了辛酸的歷史

It took 4 years after Maxwell equations to demonstrate E&Mwaves. It took 29 years to realize that they implied synchrotron radiation, and another 36 years to observe it!

CHAPTER 1
Radiation from a point charge

Electromagnetic radiation emitted from charged particles when they are accelerated is called *synchrotron radiation*.

Consider a point charge e following a *prescribed motion* $\vec{r}(t)$.
重要觀念！運動是“事先設定”的

- What is a prescribed motion? Answer: we are given $\vec{r}(t)$. Once $\vec{r}(t)$ is given, $\vec{\beta}(t)$ and $\vec{\beta}(t)$ are known.
- Once $\vec{r}(t)$, $\vec{\beta}(t)$, $\vec{\beta}(t)$ are known, we calculate *everything* about the synchrotron radiation of the particle.
- It does not matter whether $\vec{r}(t)$ is due to gravitational, electrical, magnetic, nuclear, or any combination of these forces.

Detailed derivations are omitted below. Find them in electromagnetism textbooks.

Synchrotron radiation is:

more pronounced for $\cdots$	less pronounced for $\cdots$
electrons	protons
relativistic particles	nonrelativistic particles
transverse acceleration	longitudinal acceleration
forward direction	nonforward direction
high frequency radiation	low frequency radiation

記住這張表。We will prove each one of these features in the following text.

The features that radiation

(a) is highly collimated in the forward direction

(b) extends to very high frequencies

provide the basis of many applications of synchrotron radiation today.

Radiation power

Given the prescribed motion $\vec{r}(t)$, the instantaneous differential power per unit solid angle radiated by the point charge is

$$\frac{dP(t)}{d\Omega} = \frac{r_0 mc}{4\pi}\frac{1}{\kappa^5}\left|n \times [(n - \vec{\beta}) \times \dot{\vec{\beta}}]\right|^2 \tag{1}$$

推導省略

$$\begin{aligned}
n &= \text{observation direction} \\
r_0 &= \text{classical radius} = \frac{e^2}{4\pi\epsilon_0 mc^2} = \begin{cases} 2.82 \times 10^{-15}\,\text{m,electron} \\ 1.53 \times 10^{-18}\,\text{m,proton} \end{cases} \\
\kappa &= \text{``Dopplerfactor''} = 1 - \beta\cos\theta \\
\theta &= \text{angle between } n \text{ and } \vec{\beta}
\end{aligned}$$

- Eq.(1) is valid for arbitrary $\vec{r}(t)$. Motion can be nonrelativistic or relativistic.

- No radiation if $\vec{\beta}(t) = \text{constant}$.

- The factor $1/\kappa^5$ is important $\Longrightarrow$ extremely sensitive when $\kappa \ll 1$.

- When does $\kappa \ll 1$? Answer: when $\beta \to 1$ and $\theta \to 0$, i.e. when relativistic, and when radiating in the forward direction. We then have

$$\frac{1}{\kappa^5} \approx \frac{32}{\left(\theta^2 + \frac{1}{\gamma^2}\right)^5} \tag{2}$$

- Eq.(2) $\Longrightarrow$ Radiation is sharply peaked in the forward direction $\theta = 0$ with

$$\text{opening angle} \sim \frac{1}{\gamma}$$

γ 越大，同步幅射越朝前。

With $\theta \lesssim \frac{1}{\gamma}$, we have $1/\kappa^5 \sim \gamma^{10}$.

We can integrate Eq.(1) over the solid angle $\int d\Omega$ to obtain

$$P = \frac{2}{3} r_0 mc\gamma^2 \left[\left(\frac{d\gamma\vec{\beta}}{dt} \right)^2 - \left(\frac{d\gamma}{dt} \right)^2 \right] \tag{3}$$

Homework 1 Prove Eq.(2) for a relativistic point charge with $\gamma = \frac{1}{\sqrt{1-\beta^2}} \gg 1$ and $\theta \ll 1$.

Homework 2 Use Eq.(3) to show that a particle in uniform circular motion with radius ρ radiates a power

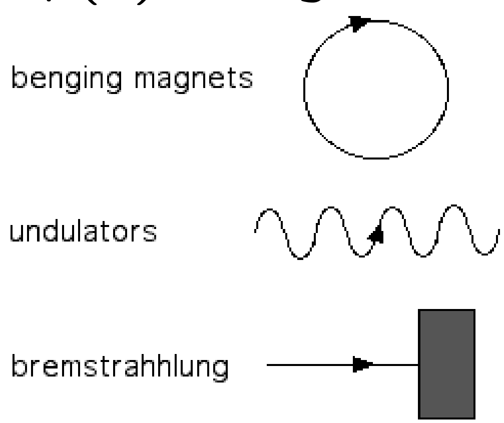
$$P = \frac{2}{3} r_0 m c^3 \frac{\beta^4 \gamma^4}{\rho^2} \tag{4}$$

Radiation spectrum

Given the prescribed motion, one can also calculate the differential radiation spectrum per solid angle:

$$\frac{d^2W}{d\omega d\Omega} = \frac{r_0 mc}{4\pi^2}\omega^2 \left| \int_{-\infty}^{\infty} dt\, e^{-i\omega\left(t-\frac{n\cdot\vec{r}}{c}\right)} \left[n \times \left(n \times \vec{\beta}\right)\right] \right|^2 \quad \text{推導省略} \tag{5}$$

- We have changed our attention from radiated power to radiated energy!
- Eq.(5) is a general expression, applicable to



Bending magnet radiation

Apply Eq.(5) to a point charge of energy γmc^2 moving along a circular trajectory with curvatue ρ,

$$\frac{d^2W}{d\omega d\Omega} = \frac{3r_0mc^2}{4\pi^2c}\gamma^2\left(\frac{\omega}{\omega_c}\right)^2(1+\gamma^2\theta^2)^2\left[K^2_{2/3}(\xi)+\frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}K^2_{1/3}(\xi)\right] \quad (6)$$

推導省略

$$\begin{aligned}
\theta &= \text{angle between } n \text{ and bending plane} \\
&\quad (\neq \text{ angle between } n \text{ and } \vec{\beta}\ ) \\
\xi &= \frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{3/2} \\
\omega_c &= \text{critical frequency} = \frac{3}{2}\frac{\gamma^3c}{\rho} \\
K_{2/3,\,1/3}(\xi) &= \text{Bessel functions}
\end{aligned}$$

Comments concerning Eq.(6):

- $\theta =$ angle normal to the bending plane 上下張角, $\psi =$ bending angle. Uniform bending $\Longrightarrow$ Eq.(6) depends on θ but not on ψ.

- The solid angle element

$$d\Omega = \cos\theta d\theta d\psi \approx d\theta d\psi$$

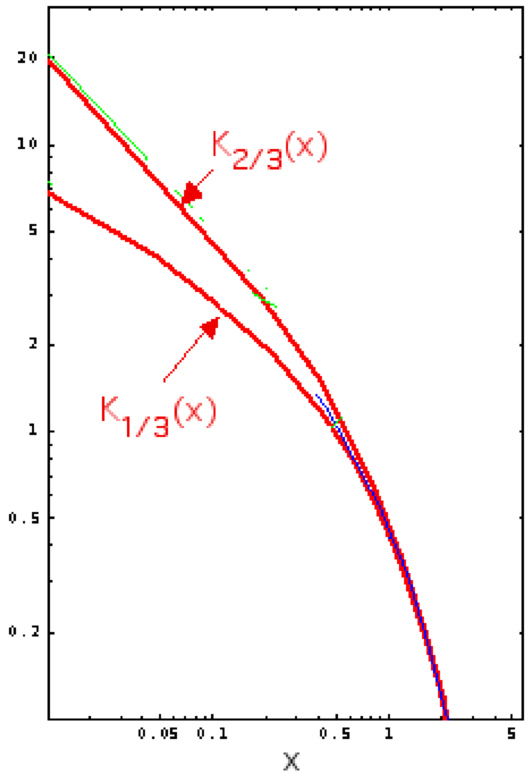
- θ always appears in a combined quantity $\gamma\theta$ $\Longrightarrow$ angular distribution in θ simply scales with $1/\gamma$.

Polarization

Synchrotron radiation is highly polarized: components parallel (σ-mode) and normal (π-mode) to the bending plane.

同步幅射光是高度（~87.5%）極化的。

The first term in Eq.(6) gives the energy radiated into the σ-mode polarization. The second term gives the π-mode.



From the Bessel functions, it follows that most radiation occurs when

$$\xi \lesssim 1$$

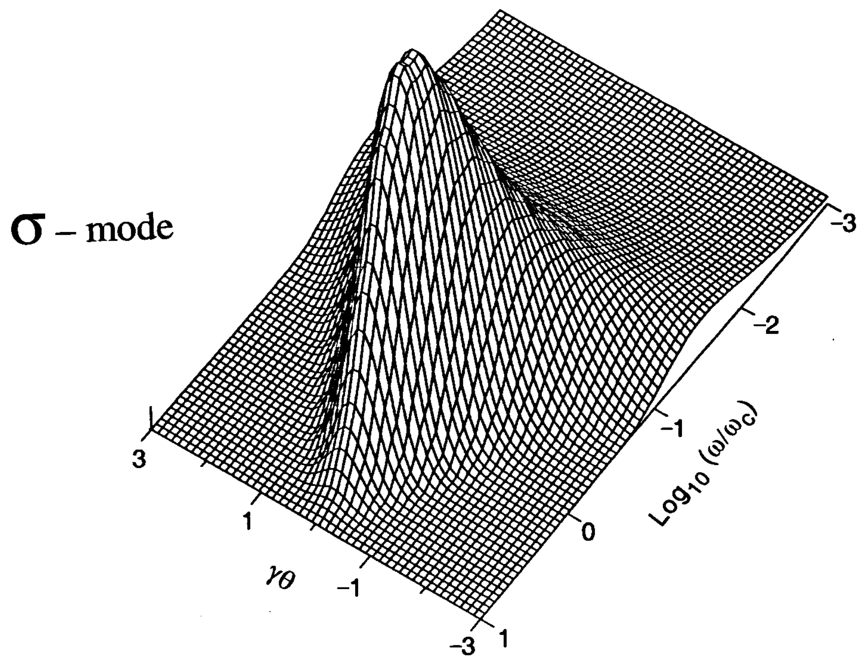
or

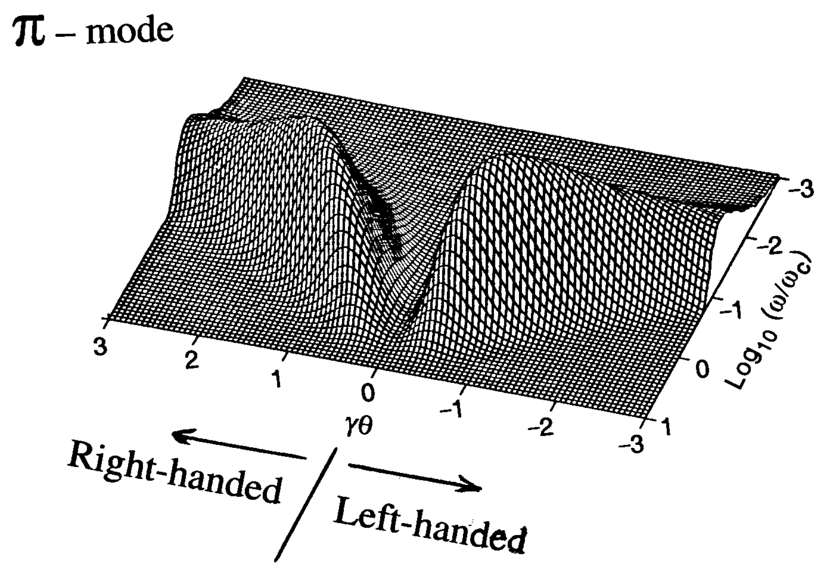
$$\omega \lesssim \frac{2\omega_c}{(1+\theta^2\gamma^2)^{3/2}} \quad \text{at fixed } \theta$$

or

$$\theta \lesssim \left(\frac{\omega_c}{\omega}\right)^{1/3} \frac{1}{\gamma} \quad \text{at fixed } \omega$$

We said before that radiation occurs with an opening angle $\sim \frac{1}{\gamma}$.
But that is true only on average. The truth is:
radiation $\omega > \omega_c$ has $\theta < \frac{1}{\gamma}$,
radiation $\omega < \omega_c$ has $\theta > \frac{1}{\gamma}$.


σ – mode
γθ
3
1
−1
−3
Log10 (ω/ωc)
1
0
−1
−2
−3


π – mode
γθ
3
2
1
0
−1
−2
−3
Log10 (ω/ωc)
1
0
−1
−2
−3
Right-handed
Left-handed

- The total volume in the upper figure is 7 times the lower figure. Total radiation is 87.5% polarized.
- σ-mode has a maximum radiation at $\theta = 0$, while the π-mode vanishes. Radiation is 100% polarized at $\theta = 0$.
- Along $\omega = \omega_c$, the distribution has a width $\gamma\theta \approx \pm 1$. As ω changes, the width changes as $\approx \pm(\omega/\omega_c)^{1/3}$. This is true for both polarizations.

Homework 3 A photon that carries the critical frequency has a critical photon energy $u_c = \hbar\omega_c$. Show that

$$u_c\,[\mathrm{keV}] = 2.22\frac{E^3\,[\mathrm{GeV}^3]}{\rho\,[\mathrm{m}]} = 0.665E^2\,[\mathrm{GeV}^2]\,B\,[\mathrm{T}]$$

提示： 把 ω_c 轉換成 u_c

把 γ 轉換成 E

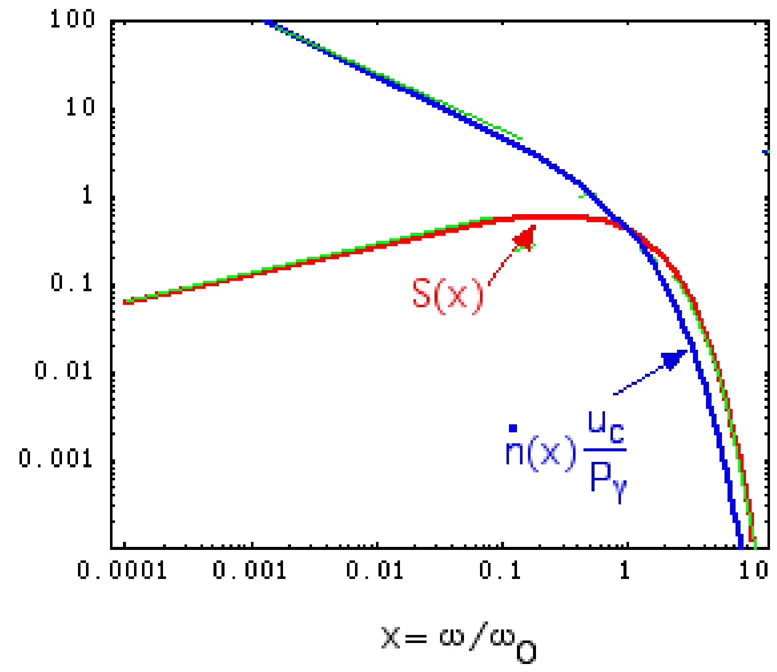
把 ρ 轉換成 B

Frequency spectrum

Integrate $\frac{d^2W}{d\omega d\Omega}$ over θ 對 θ 積分,

$$\frac{d^2W}{d\omega d\psi} = \frac{4}{9} r_0 mc\gamma \; S\left(\frac{\omega}{\omega_c}\right) \tag{7}$$

$$S\left(\frac{\omega}{\omega_c}\right) = \frac{9\sqrt{3}}{8\pi}\frac{\omega}{\omega_c}\int_{\omega/\omega_c}^{\infty} dx\, K_{5/3}(x)$$



- Distribution in ψ is uniform. Eq.(7) is independent of ψ.
- Also plotted is $n(x)$ (see later).
- Normalization:

$$\int_0^\infty dx\, S(x) = 1$$

$$\int_0^\infty dx\, n(x) = \frac{15\sqrt{3}}{8\pi}$$

- Exactly half of the radiation has $\omega < \omega_c$, while half has $\omega > \omega_c$,

$$\int_0^1 dx\, S(x) = \int_1^\infty dx\, S(x) = \frac{1}{2}$$

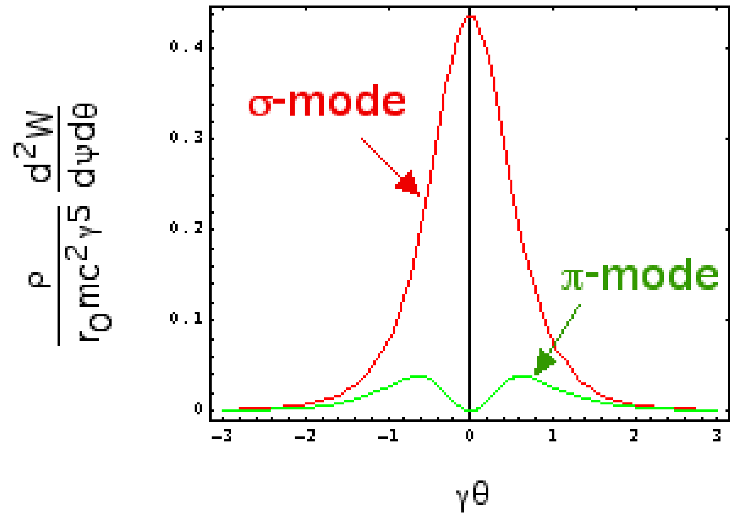
Homework 4 Write a small numerical program to show that about 91% of the photons have $u < u_c$, while 9% have $u > u_c$.

Angular distribution

Integrate $\frac{d^2W}{d\omega d\Omega}$ over ω 對 ω 積分,

$$\frac{d^2W}{d\psi d\theta} = \frac{1}{16} \frac{r_0 mc^2}{(1+\gamma^2\theta^2)^{5/2}} \left(7 + \frac{5\gamma^2\theta^2}{1+\gamma^2\theta^2}\right) \frac{\gamma^5}{\rho} \tag{8}$$

The two terms are the σ- and the π-modes. The π-mode vanishes along $\theta = 0$.



Radiation power

Integrate $\frac{d^2W}{d\omega d\Omega}$ over both ω and θ 對 θ 和 ω 積分,

$$\frac{dW}{d\psi} = \frac{r_0 mc^2\gamma^4}{12\rho}(7+1) \tag{9}$$

- The point charge radiates 7 times more energy into the σ-mode than into the π-mode.
- Total radiated energy per bending angle:

$$\frac{dW}{d\psi} = \frac{2r_0 mc^2\gamma^4}{3\rho}$$

- Total power:

$$P_\gamma = \frac{dW}{d\psi}\frac{c}{\rho} = \frac{2r_0 mc^3\gamma^4}{3\rho^2}$$

- Total energy radiated per revolution:

$$\begin{aligned} U_0 &= 2\pi\frac{dW}{d\psi} = \frac{4\pi r_0 mc^2\gamma^4}{3\rho} \\ &= \begin{cases} 0.0885\,[\mathrm{MeV}]\ \dfrac{(E[\mathrm{GeV}])^4}{\rho[\mathrm{m}]} & \text{for electrons} \\ 0.00778\,[\mathrm{MeV}]\ \dfrac{(E[\mathrm{TeV}])^4}{\rho[\mathrm{m}]} & \text{for protons} \end{cases} \end{aligned}$$

- 前幾頁公式太複雜，不看？
  至少把這兩頁看了！

Homework 5 Define $C_\gamma \equiv \frac{4\pi}{3}\frac{r_0}{(mc^2)^3}$.

(a) Show that

$$P_\gamma = C_\gamma \frac{cE^4}{2\pi\rho^2}, \quad \text{and} \quad U_0 = C_\gamma \frac{E^4}{\rho}$$

(b) Note the strong dependence E^4. Synchrotron radiation is strongly a relativistic effect. Note $C_\gamma \propto m^{-4}$. This means protons radiate less than electrons by a factor of $\left(\frac{m_e}{m_p}\right)^4 = 8.80 \times 10^{-14}$.

(c) Find ρ in terms of the bending magnetic field B, and show

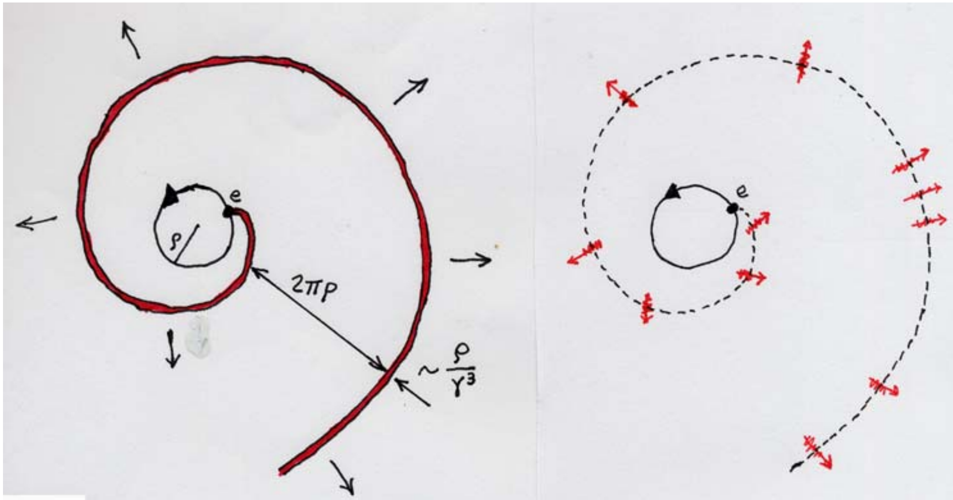
$$P_\gamma = C_\gamma \frac{e^2c^3}{2\pi\beta^2}E^2B^2, \quad \text{and} \quad U_0 = C_\gamma \frac{ec}{\beta}E^3B$$

Quantum fluctuations

There are two pictures of synchtrotron radiation:

Picture	Radiation is in the form of $\cdots$	Emission occurs $\cdots$
Classical	electromagnetic wave	continuously, 100% of the time
Quantum mechanical	quantized photons	discretely, only at 2% of the time

These two pictures are fundamentally different. Nature told us the correct picture is quantum mechanical!



In the quantum mechanical picture,

- time of each photon emission is random
- amount of energy of each photon is random

All we know is that, *on the average*, the radiated spectrum is Eq.(7).

Let $n(u)du$ be the average number of photons emitted per electron per unit time in the energy range between u and $u + du$, then

$$un(u)du \;=\; \frac{c}{\rho}\frac{d^2W}{d\omega d\psi}d\omega \;=\; \frac{4\,r_0 mc^2\gamma}{9\,\rho}\;S\left(\frac{\omega}{\omega_c}\right)d\omega$$

$$\Longrightarrow\;\; n(u) \;=\; \frac{P_\gamma}{u_c^2}\frac{u_c}{u}S\left(\frac{u}{u_c}\right) \tag{10}$$

where $u_c = \hbar\omega_c$.

Total number (on average) of photons radiated per unit time:

$$\mathcal{N} = \int_0^\infty n(u)du = \frac{5\alpha c\gamma}{2\sqrt{3}\,\rho} \tag{11}$$

where $\alpha \approx 1/137$ is the fine structure constant.

Average number of photons emitted per turn:

$$\mathcal{N}_0 = \mathcal{N}\frac{2\pi\rho}{c} = \frac{5\pi\alpha\gamma}{\sqrt{3}} \approx \frac{\gamma}{15}$$

This is a surprisingly simple expression! For example, a 2-GeV particle radiates ~ 250 photons per turn. It is independent of

- bending radius,
- type of particle!

Another useful quantity (see later) is

$$\mathcal{N}\langle u^2\rangle = \int_0^\infty u^2 n(u)du = \frac{55}{24\sqrt{3}}\alpha\hbar^2 c^3\frac{\gamma^7}{\rho^3} \tag{12}$$

Note the 7-th power of γ! The $\hbar$ indicates its quantum mechanical nature.

Homework 6 Gain insight by going through the derivation of Eq.(10).

Homework 7 What is the beam energy of the storage ring in the figure on p.21 illustrating the quantum mechanical picture?

Homework 8 A 2-GeV electron radiates ~ 250 photons per turn. So does a 2-GeV proton. So why did we insist that synchrotron radiation is more pronounced for electrons than protons?

Each emission is a statistically independent event

Each emission occurs over a small piece of arc of particle s trajectory. The arc subtends an angle $\sim \pm\frac{1}{\gamma}$.

$\Longrightarrow$

The total bending angle of emission is

$$\sim \frac{\gamma}{15} \times \frac{2}{\gamma} = \frac{2}{15} \text{ radians/turn}$$

which is simply a constant!

$\Longrightarrow$

An accelerated particle radiates over only $\frac{2/15}{2\pi} \approx 2\%$ of its bent trajectory. It does *nothing* over the remaining 98%! Radiation events are sparse. Duty factor is only 2%.

$\Longrightarrow$

Conclusion:

- Each emission can be considered an independent event.
- Synchrotron radiation events can be considered a statistical process.
- 光子是一個一個放射出來的。
  同步幅射是萊福槍，不是機關搶。

This conclusion is independent of

- bending radius,
- type of particle,
- and even γ!

CHAPTER 2
What happens to the point charge when it radiates?

So far, we assumed prescribed motion. Radiation does *not* act back on the particle:

- The particle does not receive reduction in energy or a recoil in momentum as it radiates $\Longrightarrow$ Energy and momentum are not conserved! 連能量和動量守恆都不管了
- Quantum mechanical picture
  $\Longrightarrow$ Radiation is discrete and random
  $\Longrightarrow$ noise! 只有量子圖象中才出現
  This noise has been ignored.

These are OK for a synchrotron radiation user, but not for accelerator physicists, who care about what happens to the particle.

One obvious effect due to energy conservation is that the lost energy must be replenished by the RF cavity. But we must consider three additional effects:

1. radiation damping
2. quantum excitation
3. quantum lifetime

We take a perturbative approach:

- Step 1: Calculate radiation assuming unperturbed prescribed motion.
- Step 2: Use the radiation obtained from Step 1 to calculate its effects on particle motion.

- There is no Step 3. We stop here. The motion is now perturbed from the prescribed, but we do not re-calculate the radiation.

We will show the following:

	Physical origin	Picture
Radiation damping	Momentum conservation	Classical
Quantum excitation	Energy conservation	Quantum mechanical

Radiation damping --- rough calculations

There are three oscillations of a particle in a storage ring: horizontal betatron oscillation (x-motion), vertical betatron oscillation (y-motion), and synchrotorn oscillation (z- or δ-motion). We are very lucky that all three oscillations are *damped* by synchrotron radiation. 謝天謝地 We will calculate the three damping times τ_x, τ_y, τ_z.

Calculate τ_z

z-motion and δ-motion are interconnected by synchrotron oscillation, where $\delta = \frac{\Delta E}{E}$. z 和 δ 是綁在一起的 To calculate τ_z, we calculate the damping on δ due to synchrotron radiation.

Consider a particle with energy $E_0 + \Delta E$:

$$P_\gamma(\Delta E) = C_\gamma \frac{e^2 c^3}{2\pi\beta^2}(E_0 + \Delta E)^2 B^2$$

$$\Longrightarrow$$

$$\text{change of } \Delta E \text{ per turn} = -[P_\gamma(\Delta E) - P_\gamma(0)]T_0$$

$$= -2\frac{P_\gamma(0)\Delta E}{E_0}T_0 = -2\frac{U_0}{E_0}\Delta E$$

This gives a damping effect on ΔE. The origin of damping is that higher energy particles radiate more.

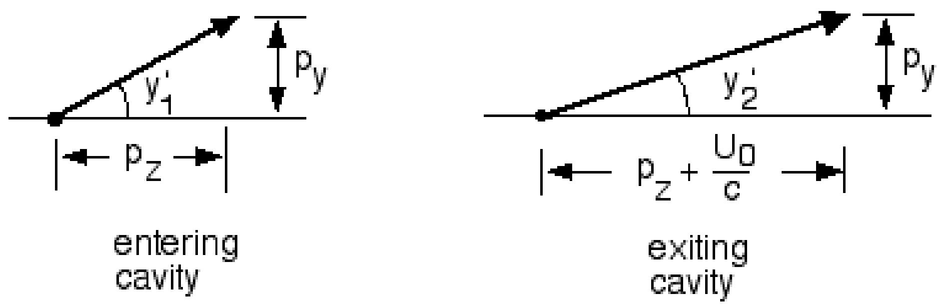
The damping decrement is $\frac{2U_0}{E_0}$ per turn. The damping time is

$$\tau_z = \frac{2T_0}{\text{damping decrement}} = \frac{E_0}{U_0}T_0$$

Calculate τ_x, τ_y

Photons are emitted in the instantaneous direction of motion. Each time it radiates, its momentum vector becomes shorter, but its direction is unchanged $\Longrightarrow$ x, x', y, y' are unchanged $\Longrightarrow$ Emissions do not damp the betatron amplitudes of the electrons.

However, the electron s momentum needs to be replenished at the RF cavity by U_0/c each turn. But the RF acceleration is along the z-axis $\Longrightarrow$ the electron s slopes x' and y' will be reduced (assume $\frac{U_0}{c} \ll p_z$):



$$y_2' = \frac{p_y}{p_z + \frac{U_0}{c}} \approx \frac{p_y}{p_z}\left(1 - \frac{U_0}{p_z c}\right) = y_1'\left(1 - \frac{U_0}{p_z c}\right) \approx y_1'\left(1 - \frac{U_0}{E_0}\right)$$

$$x_2' \approx x_1'\left(1 - \frac{U_0}{E_0}\right)$$

Both betatron oscillations have damping decrement per turn $\frac{U_0}{E_0}$,

$$\tau_x = \tau_y = \frac{2E_0}{U_0} T_0$$

- No $\hbar$ s $\Longrightarrow$ Radiation damping is a classical effect.
- In addition to a difference of a factor of 2 in the damping rates: Synchrotron damping occurs at bending magnets, while betatron dampings occur at the RF cavity. 物理根源完全不同
- All three radiation damping times $\sim E_0/U_0$ turns. Although $U_0 \ll E_0$, the damping rates are very fast! 再次感謝 If $E_0/U_0 \sim 10^3$, the three damping rates $\sim$1000 turns.

Radiation damping --- detailed calculations

The previous rough calculations require a correction:

$$
\begin{aligned}
\tau_x &= \frac{2E_0T_0}{U_0(1-\mathcal{D})} \\
\tau_y &= \frac{2E_0T_0}{U_0} \\
\tau_z &= \frac{E_0T_0}{U_0(0)(1+\frac{\mathcal{D}}{2})}
\end{aligned}
\tag{13}
$$

The correction is represented by a dimensionless *radiation damping partition number*,

$$
\mathcal{D} = \frac{\oint \frac{D\,ds}{\rho}\left(\frac{2G}{B\rho} + \frac{1}{\rho^2}\right)}{\oint \frac{ds}{\rho^2}} \tag{14}
$$

D 和 $\mathcal{D}$ 别弄混了！The first term in $\mathcal{D}$ occurs in combined-function magnets with dispersion. The second term occurs in dipoles with dispersion.

- $\mathcal{D}$ is a property of the lattice design, independent of beam properties.

- Expressions for $\mathcal{D}$:

$$\mathcal{D} \begin{cases} = 1, & \text{uniform dipole} \\ \approx \frac{1}{\nu_x^2} \ll 1, & \text{strong focusing, separated-function} \\ \approx 2, & \text{strong focusing, combined-function} \\ = \frac{1-2n}{1-n}, & \text{weak focusing} \end{cases} \tag{15}$$

($\nu_x =$ horizontal betatron tune)

- Usually $\mathcal{D} > 0$ $\Longrightarrow$ these correction terms cause the horizontal betatron amplitude to grow and synchrotron oscillation to damp. It is really a *synchro-betatron coupling* effect, not a true damping.

- For both x and z to be damped, we need

$$1 > \mathcal{D} > -2$$

  $1 > \mathcal{D}$ is for stability of x; $\mathcal{D} > -2$ is for stability of z.

- All modern synchrotrons are strong focusing, separate-function $\Longrightarrow$ $\mathcal{D} \approx 0$ $\Longrightarrow$ Rough calculations are valid!

Homework 9

(a) Use Eq.(14) to show $\mathcal{D} = 1$ for a storage ring made of a single uniform dipole magnet.

(b) The BNL AGS and the Fermilab Booster are strong focusing, combined-function synchrotrons. Use Eq.(15) to show that we can not store electrons in them. Otherwise, which dimension is the beam unstable? How long do you think the beam can stay in them?

Orlov-Robinson-Tarasov sum rule

A general theorem can be derived using fundamental matrix beam dynamics:

$$\sum_{k=x,y,z} \tau_k^{-1} = 2\frac{U_0}{E_0 T_0}$$

Eq.(13) holds for an unperturbed planar lattice. It satisfies this sum rule. However, even when the dynamics is fully coupled in 6-D phase space, the sum rule still holds.

Quantum excitation

According to the picture so far, a stored beam will shrink to an absolute zero size. 荒謬的結論 We must ask what effects will emerge as the beam size gets smaller:

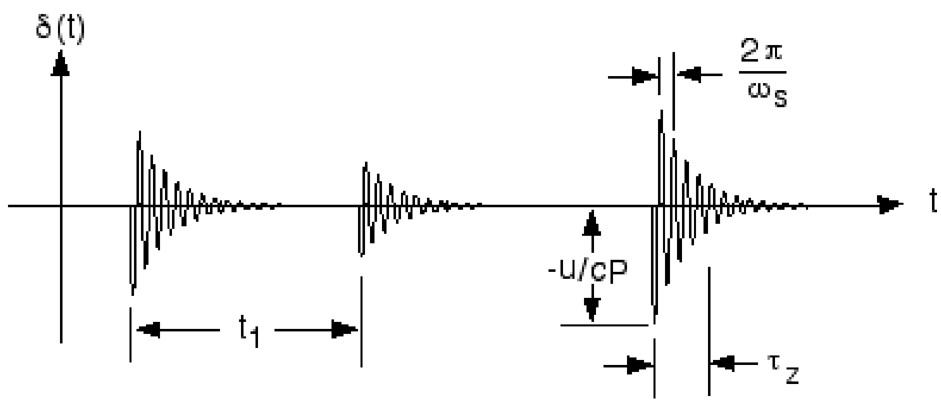
- space charge effects
- intrabeam scattering
- collective instabilities for high intensity beams
- beam-beam interaction
- magnet power supply ripples
- ground vibration
- quantum excitation

這麼多可能，你覺得那一項最大？

The answer is quantum excitation. The surprise is that (a) what limits the shrinking is the same mechanism for the shrinking, i.e. synchrotron radiation itself, and (b) this is true even when quantum excitation effect is proportional to $\hbar$, and is supposed to be small. Quantum mechanics dominates over all the other classical effects!

Quantum excitation of synchrotron oscillations

Consider a synchronous particle that at time $t = 0$ emits a photon with energy u. Its energy gets a recoil $-u$, and begins to execute a synchrotron oscillation with frequency ω_s. Because of radiation damping, this oscillation damps out in time τ_z. At time $t = t_1$ later, this electron emits another photon, producing another excitation-damping cycle. The process continues.



Photon energy of each emission is random, but all $\sim u_c$. Emission times are random, but with an average

$$t_1 \sim \frac{1}{\mathcal{N}} = \frac{T_0}{\mathcal{N}_0} = \frac{T_0}{\gamma/15}$$

t_1 is extremely short $\Longrightarrow$ the excitation-damping cycles overlap grossly, and not as cleanly separated as in this figure.

There is a hierarchy of time scales in an electron storage ring 重要觀念 !:

$$\tau_{\mathrm{pol}} \gg \tau_{x,y,z} \gg \frac{2\pi}{\omega_s} \gg T_0 \gg \frac{2\pi}{\omega_{x,y}} \gg t_1$$

where τ_{pol} is spin polarization time.

The particle is constantly excited and damped. An equilibrium is reached. The particle acquires an rms equilibrium energy spread from a balance between quantum excitation and radiation damping,

$$\delta(t) = \sum_{i=-\infty_{(t>t_i)}}^{\infty} \left(\frac{-u_i}{E_0}\right) e^{-(t-t_i)/\tau_z} \cos\omega_s(t-t_i)$$

$$\Longrightarrow$$

$$\begin{aligned} \sigma_\delta^2 &= \langle \delta^2(t) \rangle \\ &= \left\langle \sum_{i,j}{}_{(t>t_i, t>t_j)} \left(\frac{u_i u_j}{E_0^2}\right) e^{-(t-t_i)/\tau_z} \cos\omega_s(t-t_i) e^{-(t-t_j)/\tau_z} \cos\omega_s(t-t_j) \right\rangle \end{aligned}$$

where $\langle\ \rangle$ is averaging over photon emission probabilities.

$$\langle \cos\omega_s(t-t_i)\cos\omega_s(t-t_i) \rangle \to \frac{1}{2}\delta_{ij}$$

$$\left\langle \sum_{i} {}_{(t>t_i)} e^{-2(t-t_i)/\tau_z} \right\rangle \rightarrow \mathcal{N} \int_{-\infty}^{t} dt_i e^{-2(t-t_i)/\tau_z} = \mathcal{N}\frac{\tau_z}{2}$$

$$\Longrightarrow$$

$$\sigma_\delta^2 = \frac{\mathcal{N}\langle u^2\rangle \tau_z}{4E_0^2}$$

where $\mathcal{N}\langle u^2\rangle$ is given by Eq.(12):

$$\begin{aligned} \sigma_\delta^2 &= \frac{55}{32\sqrt{3}}\frac{\hbar}{mc}\,\frac{\gamma^2}{2+\mathcal{D}}\,\frac{\oint \frac{ds}{|\rho|^3}}{\oint \frac{ds}{\rho^2}} \\ &= \left[3.83\times 10^{-13}\,\mathrm{m}\right]\,\frac{\gamma^2}{2+\mathcal{D}}\,\frac{\oint \frac{ds}{|\rho|^3}}{\oint \frac{ds}{\rho^2}} \qquad (16) \end{aligned}$$

where we have averaged over the circumference. Note an absolute value for $\rho(s)$. The numerator $\oint \frac{ds}{|\rho|^3}$ comes from quantum excitation. The denominator $\oint \frac{ds}{\rho^2}$ comes from radiation damping. Their ratio indicates a balancing between them.

Although proportional to $\hbar$, σ_δ^2 is also proportional to γ^7. For relativistic beams, the rms energy spread *can* be macroscopic.

For an isomagnetic ring,

$$\sigma_\delta^2 = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \frac{\gamma^2}{(2+\mathcal{D})\rho}$$

Take $E_0 = 5$ GeV, $\rho = 30$ m, $\mathcal{D} = 0$, we find $\sigma_\delta = 0.8 \times 10^{-3}$.

- In a given a storage ring (ρ fixed), we find

$$\sigma_\delta \propto \gamma \qquad (\gamma \leq \text{maximum design energy } \gamma)$$

- To design an electron storage ring for a given peak energy γ, we should choose

$$\rho \propto \gamma^2, \quad \text{i.e.} \quad \text{Ring size} \propto (\text{Design energy})^2$$

High energy rings become very expensive.

- We are dealing with *longitudinal* dynamics, but σ_δ is independent of the RF system!

Homework 10 Verify (16) following the steps in the text.

Homework 11 σ_δ can be expressed in terms of E_0 and u_c.
(a) Find this expression.
(b) What is σ_E when $E_0 = 10$ GeV and $u_c = 20$ KeV?

Solution (a) $\sigma_\delta^2 \approx \frac{55}{192\sqrt{3}} \frac{u_c}{E_0}$

Quantum excitation of horizontal betatron oscillations

When a photon of energy u is emitted, the electron energy changes by $-u$. After the emission, the equilibrium orbit of the particle s horizontal betatron oscillation is suddenly shifted by $-D\frac{u}{E_0}$, where D is dispersion function.

The horizontal betatron amplitude

$$A_x^2 = \frac{x_\beta^2}{\beta_x} + \frac{(\beta_x x'_\beta + \alpha_x x_\beta)^2}{\beta_x}$$

is changed by an amount

$$\Delta A_x^2 = \frac{u^2}{E_0^2}\mathcal{H}, \qquad \mathcal{H} = \frac{D^2 + (\beta_x D' + \alpha_x D)^2}{\beta_x}$$

Quantum excitation comes from dispersion D and D'. Note: no dispersion in vertical.

Properties of $\mathcal{H}$:

- a lattice property, and is a function of s.
- dimensionality of length.
- always positive.
- contains two terms. Usually the first term dominates. Roughly,

$$\mathcal{H} \approx \frac{D^2}{\beta_x} \approx \frac{(R/\nu_x^2)^2}{R/\nu_x} = \frac{R}{\nu_x^3}$$

There is a hierarchy of lengths in a storage ring 重要觀念 !:

$$
\begin{aligned}
& R \\
& \beta_x \approx \frac{R}{\nu_x} \\
& D \approx \frac{R}{\nu_x^2} \\
& \mathcal{H} \approx \frac{R}{\nu_x^3}
\end{aligned}
$$

$$R \;\gg\; \beta_{x,y} \;\gg\; D \;\gg\; \mathcal{H}$$

- Its physical meaning is a *coupling coefficient* from energy recoil to horizontal betatron oscillation. This coupling is the smaller the better.

The rate of change of the beam average $\langle A_x^2 \rangle$ is

$$\frac{d}{dt}\langle A_x^2 \rangle = -\frac{2}{\tau_x}\langle A_x^2 \rangle + \frac{1}{2\pi R E_0^2}\oint ds\, \mathcal{H}\mathcal{N}\langle u^2 \rangle \tag{17}$$

First term is radiation damping. Second term is quantum excitation, averaged over the circumference $2\pi R$.

Eq.(17) determines the time evolution of the beam emittance after injection. Equilibrium is reached when

$$\frac{d}{dt}\langle A_x^2 \rangle = 0$$

$$\Longrightarrow \quad \langle A_x^2 \rangle_{\text{eq}} = \frac{\tau_x}{4\pi R E_0^2}\oint ds\, \mathcal{H}\mathcal{N}\langle u^2 \rangle$$

$$\Longrightarrow \quad \frac{\sigma_{x\beta}^2}{\beta_x} = \frac{\langle A_x^2 \rangle_{\text{eq}}}{2} \; = \; \frac{\tau_x}{8\pi R E_0^2}\oint ds\, \mathcal{H}\mathcal{N}\langle u^2 \rangle$$

$$= \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \frac{\gamma^2}{1-\mathcal{D}} \frac{\oint ds \frac{\mathcal{H}}{|\rho|^3}}{\oint \frac{ds}{\rho^2}} = \left[3.83 \times 10^{-13} \text{ m}\right] \frac{\gamma^2}{1-\mathcal{D}} \frac{\oint ds \frac{\mathcal{H}}{|\rho|^3}}{\oint \frac{ds}{\rho^2}} \tag{18}$$

The right hand side is independent of s
$\Longrightarrow \qquad \sigma_{x\beta} \propto \sqrt{\beta_x(s)}$ as a function of s.

For an isomagnetic storage ring,

$$\frac{\sigma_{x\beta}^2}{\beta_x} = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \frac{\gamma^2 \langle\mathcal{H}\rangle_{\text{dipole}}}{(1-\mathcal{D})\rho}$$

$$\langle\mathcal{H}\rangle_{\text{dipole}} = \frac{1}{2\pi\rho} \oint_{\text{dipole}} ds\,\mathcal{H}$$

If $\mathcal{H} \approx \frac{R}{\nu_x^3}$, $\mathcal{D} \approx 0$ and $\rho \approx R$, then

$$\frac{\sigma_{x\beta}^2}{\beta_x} \approx \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \frac{\gamma^2}{\nu_x^3} = \left[3.83 \times 10^{-13} \text{ m}\right] \frac{\gamma^2}{\nu_x^3}$$

In terms of σ_δ^2, we have

$$\frac{\sigma_{x\beta}^2}{\beta_x} \approx \frac{2R}{\nu_x^3}\sigma_\delta^2$$

If we further take $\beta_x \approx \frac{R}{\nu_x}$, then

$$\sigma_{x\beta} \approx \sqrt{2}\,\frac{R}{\nu_x^2}\,\sigma_\delta$$

In $R = 30$ m, $E_0 = 5$ GeV, we found earlier that $\sigma_\delta = 0.8 \times 10^{-3}$.
If $\nu_x \approx 5$, then $\sigma_{x\beta} \approx 1.3$ mm.

Total horizontal beam size

$$\sigma_x^2 = \sigma_{x\beta}^2 + D^2\sigma_\delta^2$$
$$= \frac{55}{32\sqrt{3}}\frac{\hbar}{mc}\frac{\gamma^2}{\oint \frac{ds}{\rho^2}}\left[\frac{\beta_x}{1-\mathcal{D}}\oint ds\frac{\mathcal{H}}{|\rho|^3} + \frac{D^2}{2+\mathcal{D}}\oint\frac{ds}{|\rho|^3}\right]$$

For an isomagnetic ring,

$$\sigma_x^2 = \frac{55}{32\sqrt{3}}\frac{\hbar}{mc}\frac{\gamma^2}{\rho}\left[\frac{\beta_x\mathcal{H}_{\text{dipole}}}{1-\mathcal{D}} + \frac{D^2}{2+\mathcal{D}}\right]$$

If $\beta_x \approx \frac{R}{\nu_x}$, $D \approx \frac{R}{\nu_x^2}$, $\mathcal{H} \approx \frac{R}{\nu_x^3}$, $\rho \approx R$, and $\mathcal{D} \approx 0$,

$$\sigma_x^2 = \frac{55}{32\sqrt{3}}\frac{\hbar}{mc}\frac{\gamma^2 R}{\nu_x^4}\left[1 + \frac{1}{2}\right]$$

$$= \left(3.83 \times 10^{-13} \text{ m}\right) \frac{\gamma^2 R}{\nu_x^4} \left[1 + \frac{1}{2}\right]$$

The two terms in the square brackets come from betatron and synchrotron contributions respectively.

- $\sigma_{x\beta}, \sigma_\delta$, and σ_x all are $\propto \gamma$ for a given storage ring lattice.
- $\sigma_x \propto \nu_x^{-2}$. A strongly focused lattice is a sensitive way to minimize σ_x.
- The synchrotron contribution is comparable to, but slightly less than, the betatron contribution.

FODO lattice

To provide a bright beam, we want small beam emittance. Special lattices for this purpose include the Chaseman-Green lattice (or double-bend-achromat), triple-, quadruple-bend-achromats, etc. Basically we want to minimize $\langle \mathcal{H} \rangle_{\text{dipole}}$. But here we consider a simple FODO lattice,

	at QF	at QD
$\sin\frac{\Phi}{2}$	$\frac{L}{2\lvert f\rvert}$	$\frac{L}{2\lvert f\rvert}$
β_x	$\frac{2L(1+\sin\frac{\Phi}{2})}{\sin\Phi}$	$\frac{2L(1-\sin\frac{\Phi}{2})}{\sin\Phi}$
α_x	0	0
γ_x	$\frac{1}{\beta_F}$	$\frac{1}{\beta_D}$
D	$\frac{L\theta(1+\frac{1}{2}\sin\frac{\Phi}{2})}{\sin^2\frac{\Phi}{2}}$	$\frac{L\theta(1-\frac{1}{2}\sin\frac{\Phi}{2})}{\sin^2\frac{\Phi}{2}}$
D'	0	0

$L =$ drift space length, $f =$ focal length of both the QF and QD quadrupoles, $\theta =$ dipole bend angle, and $\Phi =$ betatron phase advance per cell.

From this table, we find

$$\mathcal{H}_F = L\theta^2 \frac{\cos\frac{\Phi}{2}}{\sin^3\frac{\Phi}{2}\left(1+\sin\frac{\Phi}{2}\right)}\left(1+\frac{1}{2}\sin\frac{\Phi}{2}\right)^2$$

$$\mathcal{H}_D = L\theta^2 \frac{\cos\frac{\Phi}{2}}{\sin^3\frac{\Phi}{2}\left(1-\sin\frac{\Phi}{2}\right)}\left(1-\frac{1}{2}\sin\frac{\Phi}{2}\right)^2$$

$$\Longrightarrow \quad \langle\mathcal{H}\rangle_{\text{dipole}} \approx \frac{1}{2}(\mathcal{H}_F + \mathcal{H}_D) = L\theta^2 \frac{1-\frac{3}{4}\sin^2\frac{\Phi}{2}}{\sin^3\frac{\Phi}{2}\cos\frac{\Phi}{2}} \qquad (19)$$

Homework 12 Write a numerical program using Eq.(19) to show that a minimum of $\langle\mathcal{H}\rangle_{\text{dipole}}$, therefore minimum of emittance, occurs when $\Phi = 138^\circ$.

Quantum excitation of vertical betatron oscillations

Synchrotron radiation excites

- energy synchrotron oscillation *directly* 直接
- horizontal betatron oscillation by *coupling* to energy with coupling strength $\mathcal{H}$ 間接
- vertical betatron oscillations in three ways:
  directly, but only when photons are emitted not exactly tangentially to the direction of motion of the electron 直接
  coupling to energy but only when there is a vertical dispersion due to magnet field errors 間接
  coupling to x-motion but only by x-y coupling 間接再間接

We discuss vertical quantum excitation. All three contributions are small.

Direct contribution (the 直接 term)

Calculation gives

$$\frac{\sigma_y^2}{\beta_y} = \frac{55}{64\sqrt{3}} \frac{\hbar}{mc} \frac{\oint ds \, \frac{\beta_y}{|\rho|^3}}{\oint \frac{ds}{\rho^2}}$$ 推導省略

For an isomagnetic ring,

$$\frac{\sigma_y^2}{\beta_y} = \frac{55}{64\sqrt{3}} \frac{\hbar}{mc} \frac{\langle \beta_y \rangle_{\text{dipole}}}{\rho}$$

Let $\beta_y \approx \frac{R}{\nu_y}$ and $\rho \approx R$,

$$\sigma_y^2 \approx \frac{55}{64\sqrt{3}} \frac{\hbar}{mc} = \left(3.83 \times 10^{-13} \text{ m}\right) \frac{R}{2\nu_y^2}$$

If $R = 30$ m and $\nu_y \approx 5$, we find $\sigma_y = 0.5$ μm.

Note that this natural vertical beam size (直接 term) is for a perfect storage ring. You can never make σ_y smaller than 0.5 μm by correcting errors.
計算 coupling correction 的同學請注意！

Coupling-to-energy contribution (the 間接 term)

In reality, the vertical beam size comes mainly from errors. The 間接 term comes from spurious vertical dispersion D_y. The mechanism is similar to the horizontal quantum excitation. Roughly, just replace $\mathcal{H}_x$ by $\mathcal{H}_y$ (不完全對!).

Coupling-to-horizontal contribution (the 間接再間接 term)

In this mechanism, energy couples to horizontal by $\mathcal{H}$, and then horizontal couples to vertical by a coupling coefficient:

$$G = \frac{1}{2\pi}\oint ds \frac{1}{B\rho}\frac{\partial B_y}{\partial y}\sqrt{\beta_x\beta_y}e^{i\psi_x - i\psi_y - i(\nu_x-\nu_y-m)\frac{s}{R}}$$

G is a complex quantity, integrating skew quadrupoles $\frac{\partial B_y}{\partial y}$ around the ring. Usually we want $|G| \lesssim 0.01$.

When the betatron tunes $\nu_{x,y}$ are close to a *difference resonance* $\nu_x - \nu_y \approx m$, the quantum excitation for x will be split between x and y. Let $\epsilon_x = \sigma_{x\beta}^2/\beta_x$ and $\epsilon_y = \sigma_y^2/\beta_y$, then

$$\epsilon_x = \frac{1}{1+\kappa}\epsilon_{x0}, \qquad \epsilon_y = \frac{\kappa}{1+\kappa}\epsilon_{x0}$$

$$\kappa = \frac{|G|^2}{|G|^2 + (\nu_x - \nu_y - m)^2}$$

where ϵ_{x0} is given by Eq.(18).

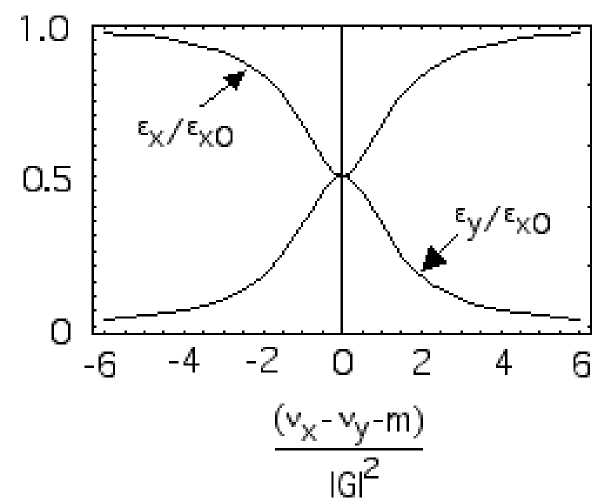
- Emittance sum rule for arbitrary coupling:

$$\epsilon_x + \epsilon_y = \epsilon_{x0}$$

- Exactly on resonance, $\kappa = 1$, then $\epsilon_x = \epsilon_y = \frac{\epsilon_{x0}}{2}$. We have a round beam.

- Far from resonance, $|\nu_x - \nu_y - m| \gg |G|$, we have $\epsilon_x = \epsilon_{x0}$ and $\epsilon_y = 0$. We have a flat beam.

- $|G|$ gives a resonance width in tune units around the resonance location $\nu_x - \nu_y - m$.



Quantum lifetimes

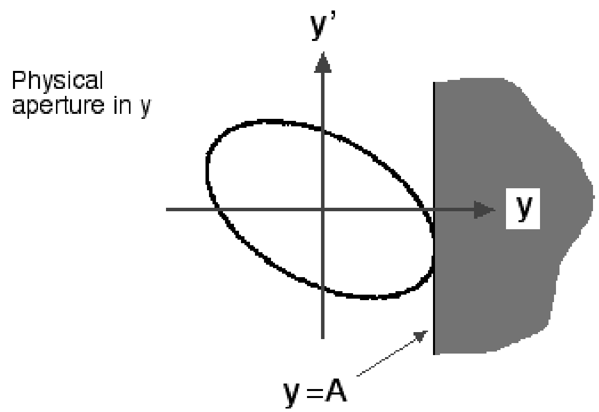
The equilibrium beam distribution in phase space is Gaussian. But this is impossible because aperture limit in vacuum chamber must cut off the Gaussian tail.

As the tail is cut off, quantum excitation will try to re-establish the tail. As particles are pumped into the tail, they are removed by the aperture. The result is: the beam will *approximately* keep a Gaussian distribution, but this entire distribution decreases slowly in time. This beam lifetime is *quantum lifetime*. There are three quantum lifetimes $\tau_{x,y,z}^{(\mathrm{q})}$, due to three aperture limits.

Vertical quantum lifetime

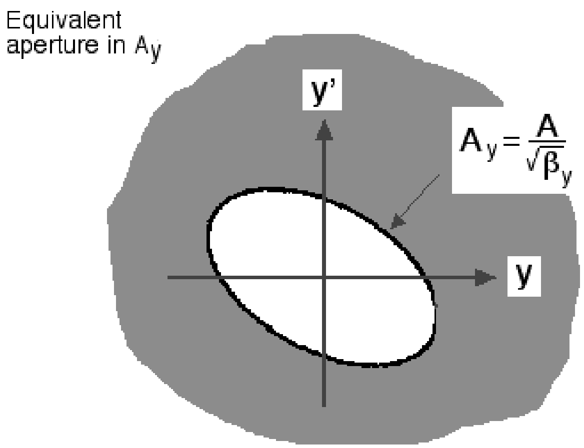
Consider a vertical aperture limit $y = A$. Whenever a particle acquires $y > A$, it is removed from the beam. Since betatron oscillation is very fast, this limit on y becomes a limit on

$$
\begin{aligned}
A_y &\le \frac{A}{\sqrt{\beta_y}} \\
A_y &= \sqrt{\frac{y^2}{\beta_y} + \frac{(\alpha_y y + \beta_y y')^2}{\beta_y}}
\end{aligned}
$$



The number of particles removed by the aperture is in the shaded region,

$$Ne^{-\frac{A^2}{2\sigma_y^2}}$$



Usually $A \gg \sigma_y \implies$ the fraction of particles removed is small.

Quantum excitation tries to re-establish beam tail. The time for this re-establishment $\sim \tau_y$. In every time interval τ_y, a fresh tail is to be re-established and subsequently removed. The particle loss rate is

$$\frac{dN}{dt} \approx \frac{N}{\tau_y} e^{-\frac{A^2}{2\sigma_y^2}} \quad \Longrightarrow \quad \tau_y^{(\mathrm{q})} \approx \tau_y e^{\frac{A^2}{2\sigma_y^2}}$$

Note that $\tau_y^{(\mathrm{q})} \gg \tau_y$.

This expression however is not accurate. The time to re-establish the beam tail is in fact not $\sim \tau_y$. τ_y is time to re-establish the *core* of the beam, but the *tail* particles move around faster by a factor of $\frac{A^2}{\sigma_y^2} \gg 1$. We really have

$$\tau_y^{(\mathrm{q})} = \left(\frac{\sigma_y}{A}\right)^2 \tau_y \, e^{\frac{A^2}{2\sigma_y^2}}$$

Quantum lifetime is extremely sensitive to aperture. With a 6-σ aperture and $\tau_y = 10$ ms, we have $\tau_y^{(\mathrm{q})} = 6$ hrs. For a 7-σ aperture, $\tau_y^{(\mathrm{q})} = 3000$ hrs.

However, we often demand an aperture $\sim$ 10-σ because:

- safety margin
- closed orbit distortion
- quantum lifetime is extremely sensitive to beam tail $\Longrightarrow$ we must save room for unavoidable nonlinearities and intra-beam scatterings

Horizontal quantum lifetime

Consider a horizontal aperture at $x = A$. If the aperture has $D = 0$, same analysis applies,

$$\tau_x^{(\mathrm{q})} = \left(\frac{\sigma_{x\beta}}{A}\right)^2 \tau_x \, e^{\frac{A^2}{2\sigma_{x\beta}^2}}$$

When $D \neq 0$, however, the aperture limit is imposed on *total* x,

$$x = x_{x\beta} + D\delta < A$$

The rms beam size is

$$\sigma_x^2 = \sigma_{x\beta}^2 + D^2\sigma_\delta^2$$

The quantum lifetime calculation involves solving Fokker-Planck diffusion equation. The result, when $r = \frac{D^2 \sigma_\delta^2}{\sigma_x^2}$ is not too close to 0 or 1, is

$$\tau^{(\mathrm{q})} = \frac{e^{n^2/2}}{\sqrt{2\pi}\, n^3} \frac{\tau_x}{(1+r)\sqrt{r(1-r)}}$$

where $n = \frac{A}{\sigma_x} \gg 1$. Compared with $D = 0$, this quantum lifetime is shorter by a factor $\sim n$.

計算 quantum lifetime 的同學請注意！

Longitudinal quantum lifetime

Synchrotron oscillation involves RF bucket dynamics. However, by analogy to the betatron cases, we have, approximately,

$$\tau_z^{(\mathrm{q})} = \left(\frac{\sigma_\delta}{\delta}\right)^2 \tau_z \, e^{\frac{\delta^2}{2\sigma_\delta^2}}$$

where δ is the momentum aperture.

δ is given by either the RF bucket half height or dynamic aperture of the storage ring lattice.